

VECTOR FIELDS ON SMOOTH THREEFOLDS VANISHING ON COMPLETE INTERSECTIONS

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1. INTRODUCTION

It is a well known fact that the existence of a vector field on a Kähler manifold with a special zero locus strongly influences the geometry of the manifold. For example, the plurigenera and some Hodge numbers vanish ([Kob72],[CL73]), and if the zero locus is projective-algebraic, the variety itself will be, too ([Hwa96]). In [Wah83] J. Wahl proved the following

Theorem 1.1. *Let X be a complex projective normal variety, L an ample line bundle, $\dim X > 1$. If $H^0(\mathcal{T}_X \otimes L^{-1}) \neq 0$ then:*

- (i) $L \cong \mathcal{O}(E)$, where the effective divisor E is a normal variety.
- (ii) $X \cong \text{Proj } A[t]$ with $A = \bigoplus_{n=0}^{\infty} H^0(E, \mathcal{O}_E(nE))$. X is the cone over E , and t has weight 1 and E is the divisor at ∞ ($t = 0$).

In particular every smooth complex projective variety with a vector field vanishing on an ample divisor is isomorphic to \mathbb{P}^N .

The aim of this paper is to look for similar statements in case of zero loci in higher codimensions, with ample normal bundle for example. The main result of this paper is in dimension 3:

Theorem 1.2. *Let X be a smooth complex projective threefold. Let D_1, D_2 be two ample effective divisors on X such that the scheme theoretic intersection $C = D_1 \cap D_2$ is an irreducible reduced curve. Let $v \in H^0(X, \mathcal{T}_X)$ be a vector field vanishing on C . Then X is isomorphic to \mathbb{P}^3 or to the three-dimensional quadric Q_3 .*

Note that the irreducibility assumption on C is necessary: Let $X \cong \mathbb{P}^2 \times \mathbb{P}^1$ and $p_1 : X \rightarrow \mathbb{P}^2, p_2 : X \rightarrow \mathbb{P}^1$ the projections, $p \in L \subset \mathbb{P}^2$ a point in a line in \mathbb{P}^2 . Let $D_1 \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the blow up of \mathbb{P}^2 in p , embedded in X . There is a point $q \in \mathbb{P}^1$ such that the fibre $p_2^{-1}(q)$ contains the strict transform of L . Let $D_2 \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the union $p_2^{-1}(q) \cup p_1^{-1}(L)$. Then D_1, D_2 are ample divisors of X and $C = D_1 \cap D_2$ is the union of two lines. Choose homogeneous coordinates $(x_0 : x_1 : x_2)$ on \mathbb{P}^2 such that $L = \{x_0 = 0\}$. Then the vector field $v = x_0 \frac{\partial}{\partial x_1} \in H^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2})$ vanishes on L and the pullback $p_1^*v \in H^0(X, \mathcal{T}_X)$ vanishes on C .

Date: February 8, 2008.

1991 Mathematics Subject Classification. Primary 14M20, Secondary 14L99, 14F05.

Key words and phrases. Characterization of the three-dimensional quadric, zero locus of vector fields.

The author gratefully acknowledges support by the DFG-Graduiertenkolleg "Komplexe Mannigfaltigkeiten" at the University of Bayreuth.

The proof of the theorem uses a modification of Wahl's theorem and Lefschetz' hyperplane theorem for an inductive argument to conclude that $\text{Pic}(X) = \mathbb{Z}$. Since there exists a vector field with nonempty zero locus X is a smooth Fano threefold. The geometry of the ample divisors discovered by Wahl's theorem rules out most of the possibilities in Iskhovskhi's classification ([Isk77, Isk78]). The remaining cases are dealt with by looking at their vector fields.

Notation. Throughout the present work, let X be a complex projective variety. The tangent sheaf $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ is the dual of the sheaf of differentials Ω_X^1 on X . If X is smooth, \mathcal{T}_X is locally free. A vector field $v \in H^0(X, \mathcal{T}_X)$ is a global section of \mathcal{T}_X . The groups $\mathbb{C}^*, \mathbb{C}^+$ denote the multiplicative and the additive group of complex numbers.

Acknowledgement. This paper is part of a PhD thesis written while the author was a member of the DFG-Graduiertenkolleg "Komplexe Mannigfaltigkeiten" at the University of Bayreuth. The author is grateful to his advisor Th. Peternell for pointing out the problem and continuous encouragement to work on it. He would like to thank the other members of the Graduiertenkolleg, especially S. Kebekus and H. Chr. von Bothmer, for creating a stimulating atmosphere and many helpful discussions on the topic.

2. A THEOREM OF WAHL

An easy corollary of Theorem 1.1 is the following result already proven in [MS78]:

Theorem 2.1. *Let X be a complex projective smooth variety, L an ample line bundle and \mathcal{T}_X the tangent bundle. If there is a vector field $v \in H^0(X, \mathcal{T}_X)$ vanishing on an ample effective divisor $D \in H^0(X, L)$ then*

$$(X, L) = (\mathbb{P}^n, \mathcal{O}(1)) \text{ or } (\mathbb{P}^1, \mathcal{O}(2)).$$

The proof of Theorem 1.1 uses the normality of X when applying the Kodaira vanishing theorem. A thorough analysis of this proof and the ideas in [MS78] shows that one can replace normality by the existence of an effective divisor in the ample line bundle to get a slightly weaker result:

Theorem 2.2. *Let X be a complex projective variety, $\dim X > 1$, and L an ample line bundle with $H^0(X, L) \neq 0$. If $H^0(X, \mathcal{T}_X \otimes L^{-1}) \neq 0$ then there is a finitely generated graded \mathbb{C} -algebra $A \subset \bigoplus_{n=0}^{\infty} H^0(X, nL)$ and a homogeneous element $T \in H^0(X, L)$ such that*

$$A[T] = \bigoplus_{n=0}^{\infty} H^0(X, nL).$$

In this case X is isomorphic to a cone over an ample Divisor E of $\text{Proj } A$.

Proof. The last part is an easy consequence of the first statements, cf. [BS95, 5.3].

The first part can be proven exactly as in [Wah83] provided that one is still able to construct a derivation of weight -1 on $R = \bigoplus_{n=0}^{\infty} H^0(X, nL)$: By assumption there is an effective divisor $D \in H^0(X, L)$ and a vector field $v \in H^0(X, \mathcal{T}_X)$ such that $v|_D = 0$. Let $G \subset \text{Aut } X$ be the subgroup of automorphisms fixing D . The existence of v implies that G is nontrivial. Furthermore the linear representation of G on the m -jets of a point $x \in D$ is faithful if $m \gg 0$. Therefore G is linear

algebraic and contains a linear algebraic one parameter group H , i.e. $H \cong \mathbb{C}^*$ or \mathbb{C}^+ .

Since H stabilizes the divisor D there is a H -linearization of the line bundle L and a dual H -action τ on $R = \bigoplus_{n=0}^{\infty} H^0(X, L^n)$ (cf. [MF82, Prop. 1.5]). If $H = \mathbb{C}^*$ there are semi-invariant elements F_0, F_1, \dots, F_m of τ , i.e.

$$\tau(t) \cdot F_i = t^{\chi(F_i)} \cdot F_i$$

which generate R as a \mathbb{C} -algebra ($\chi(F)$ is the weight of the semi-invariant element F). Let $F_0 \in H^0(X, L)$ correspond to D .

Let $R/F_0 = \bigoplus_m H^0(D, mL|_D)$ be the homogeneous coordinate ring of D . Since \mathbb{C}^* acts trivial on D via τ it acts semi-trivial on $H^0(D, mL|_D)$ with weight χ_m , and the quotients $\frac{\chi_m}{m}$ are equal for all $m \geq 1$. This implies

$$s := \frac{\chi(F_1)}{\deg F_1} = \dots = \frac{\chi(F_m)}{\deg F_m},$$

and $s \neq \frac{\chi(F_0)}{\deg F_0}$ since otherwise \mathbb{C}^* would act trivial on all of X .

Now one can twist the \mathbb{C}^* -linearization: Let \mathbb{C}^* act on R via

$$\sigma(t) \cdot F = t^{-s \deg F} \cdot \tau(t)F$$

for all homogeneous $F \in R$. Hence for an arbitrary polynomial $P \in \mathbb{C}[X_0, \dots, X_m]$:

$$\sigma(t)P(F_0, F_1, \dots, F_m) = P(t^{\chi(F_0)-s}F_0, F_1, \dots, F_m),$$

and $\chi(F_0) - s \neq 0$. It follows $D_\sigma F_1 = \dots = D_\sigma F_m = 0, D_\sigma F_0 = (\chi(F_0) - s)F_0$ where D_σ is the R-derivation corresponding to σ . One can divide D_σ by F_0 to get a (-1) -derivation D_{-1} .

If $H = \mathbb{C}^+$ let F_0, F_1, \dots, F_m still be homogeneous generators of R , the element F_0 corresponding to D . Since the unipotent group \mathbb{C}^+ fixes $D = \text{Proj}(R/F_0)$ the group acts trivial on R/F_0 . This implies for homogeneous F that

$$\tau(t)F - F \in (F_0).$$

Once more one can divide the corresponding derivation D_τ by F_0 and gets a (-1) -derivation D_{-1} on R .

Now one constructs an element $t \in H^0(X, L)$ as in [Wah83, Lemma 2.7] with

$$D_{-1}t = 1.$$

Then $R \cong A[T]$ with $A = \{r \in R | D_{-1}r = 0\}$ and $\deg T = 1$ (cf. [Wah83, Prop. 2.4]), and the theorem follows. \square

3. REDUCTION TO FANO MANIFOLDS WITH PICARD NUMBER 1

As in Theorem 1.2 let D_1, D_2 be two ample effective divisors on a smooth complex projective variety X of dimension 3 such that the scheme theoretic intersection $C = D_1 \cap D_2$ is an irreducible reduced curve. This implies that D_1, D_2 are irreducible and reduced.

Let furthermore $v \in H^0(X, T_X)$ be a vector field vanishing on C .

Lemma 3.1. *If D_2 is not stabilized by v then there will be an irreducible and reduced divisor D_∞ stabilized by v , which is linearly equivalent to D_1 and whose scheme theoretic intersection with D_2 equals*

$$D_\infty \cap D_2 = D_1 \cap D_2 = C.$$

Proof. Let $G \subset \text{Aut } X$ be the connected and nontrivial algebraic subgroup of the automorphism group of X which fixes the zero locus $Z(v)$ of v . Because the action of G on the vector space of m -jets at a fixed point will be faithful for $m \gg 0$, the group G is linear algebraic.

Let $H \subset G$ be the minimal algebraic subgroup whose Lie algebra contains v . A representation of G in $GL(V)$ shows that $H \cong (\mathbb{C}^*)^k \times (\mathbb{C}^+)^l$ ([Bor91, II.7.3]). Because H is commutative, the fixed point locus X^H is contained in $Z(v)$ and H stabilizes $Z(v)$. By composing the various \mathbb{C}^* - and \mathbb{C}^+ -actions one can move the divisor D_1 along orbits to a linearly equivalent divisor D_∞ stabilized by H .

$H(Z(v)) = Z(v)$ implies $D_1 \cap D_2 \subset D_\infty \cap D_2$, linear equivalence means $D_1.D_2 = D_\infty.D_2$, consequently $D_1 \cap D_2 = D_\infty \cap D_2$. And D_∞ is reduced and irreducible because $D_\infty \cap D_2$ is. \square

Assume from now on that D_1 is stabilized by the vector field v . If $v|_{D_1} = 0$, Wahl's Theorem 1.1 will imply $X \cong \mathbb{P}^3$. If $v|_{D_1} \neq 0$, the variant Theorem 2.2 will imply that D_1 is a cone $\mathcal{C}(C, L)$ for a (possibly singular) curve C and an ample line bundle L on C .

The cone $D_1 \cong \mathcal{C}(C, L)$ is the contraction of the section of the projective space bundle $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(L))$ belonging to the projection $\mathcal{O}_C \oplus \mathcal{O}_C(L) \rightarrow \mathcal{O}_C$. Therefore, $H^2(D_1, \mathbb{Q}) = \mathbb{Q}$, and the finitely generated abelian group $H^2(D_1, \mathbb{Z})$ has rank 1. Now apply Lefschetz' hyperplane theorem: the natural map

$$H^2(X, \mathbb{Z}) \hookrightarrow H^2(D_1, \mathbb{Z})$$

is an injection. Since $\text{Num}(X)$ is a torsion free quotient of $\text{NS}(X) \subset H^2(X, \mathbb{Z})$, it follows that $\text{Num}(X) = \mathbb{Z}$. There is an ample divisor H on X and $r \in \mathbb{Z}$ such that

$$K_X \equiv rH$$

On the other hand X can be covered by rational curves: Since there is a non trivial vector field with zeroes on X one of the groups \mathbb{C}^* or \mathbb{C}^+ is acting on X (s. proof of Lemma 3.1). The closures of the orbits are rational curves. Consequently, K_X is not nef ([Kol96, II.3.13.1]), not ample, and $-K_X$ is ample. By [Isk77, Prop.1.15] (and [Sho80] for the proof of hypothesis 1.14 in [Isk77]), X is a smooth Fano threefold with

$$\text{Pic}X \cong H^2(X, \mathbb{Z}) = \mathbb{Z}.$$

A classification of these Fano threefolds is given by the following table (cf. [Isk77],[Isk78]):

| r | $(H)^3$ | $b_3/2$ | g | X |
|-----|---------|---------|-----|--|
| 4 | 1 | 0 | 33 | \mathbb{P}^3 |
| 3 | 2 | 0 | 28 | $Q \subset \mathbb{P}^4$, the quadric |
| 2 | 1 | 21 | 5 | V_1 , a covering of the cone over the Veronese surface |
| 2 | 2 | 21 | 9 | V_2 , a double covering of \mathbb{P}^3 |
| 2 | 3 | 5 | 13 | $V_3 \subset \mathbb{P}^4$, a cubic |
| 2 | 4 | 2 | 17 | $V_4 \subset \mathbb{P}^5$, an intersection of two quadrics |
| 2 | 5 | 0 | 21 | V_5 , the intersection $Gr(1, 4) \subset \mathbb{P}^9$ with \mathbb{P}^6 |
| 1 | 2 | 52 | 2 | V'_2 , a double covering of \mathbb{P}^3 |
| 1 | 4 | 30 | 3 | $V'_4 \subset \mathbb{P}^4$, a quartic |
| 1 | 4 | 30 | 3 | V''_4 , a double covering of a quadric |

| r | $(H)^3$ | $b_3/2$ | g | X |
|-----|---------|---------|-----|--|
| 1 | 6 | 20 | 4 | $V_6 \subset \mathbb{P}^5$, an intersection of a quadric with a cubic |
| 1 | 8 | 14 | 5 | $V_8 \subset \mathbb{P}^6$, an intersection of three quadrics |
| 1 | 10 | 10 | 6 | $V_{10} \subset \mathbb{P}^7$ |
| 1 | 12 | 5 | 7 | $V_{12} \subset \mathbb{P}^8$ |
| 1 | 14 | 5 | 8 | $V_{14} \subset \mathbb{P}^9$ |
| 1 | 16 | 3 | 9 | $V_{16} \subset \mathbb{P}^{10}$ |
| 1 | 18 | 2 | 10 | $V_{18} \subset \mathbb{P}^{11}$ |
| 1 | 22 | 0 | 12 | $V_{22} \subset \mathbb{P}^{13}$ |

In this table, H is the ample generator of $\text{Pic}(X)$, r is the index of V , i.e. $-K_X = r \cdot H$, and $g = -(K_X)^3/2 + 1$ is the genus of X .

The ample divisor H is very ample except when X is of type V_1, V_2, V_2', V_4'' . These cases are dealt with in the last section, while in the next section one assumes that H is very ample.

4. AN ESTIMATE FOR THE DEGREE

By assumption $D = d \cdot H$ and $D_2 = d_2 \cdot H$ are very ample divisors on X . Theorem 2.2 implies that D is a cone $\mathcal{C}(C, \mathcal{O}(D_2|_C))$ over a (possibly singular) curve C with vertex P , defined by the divisor D_2 restricted to C .

Claim 4.1. $d_2 = 1$.

Proof. By construction C is linearly equivalent to $D_{2|D}$, hence very ample. The corresponding embedding $D \hookrightarrow X \xrightarrow{|D_2|} \mathbb{P}$ maps C into a hyperplane, and the cone D consists of the lines through the cone vertex P (not in the hyperplane) and points $Q \in C$. Such a line L cuts C transversally in one point, consequently (in X):

$$1 = C.L = d_2 H.L.$$

$H.L > 0$ implies the hypothesis. □

Let $\pi : \tilde{X} \rightarrow X$ be the blow-up of X in the cone vertex P and E the exceptional divisor. Because of the universal property of the blow-up the strict transform S of D also is the blow-up of D in P . By [Ful84, 6.7.1] the effective Cartier divisor S is linearly equivalent to $\pi^*(D) - \mu_D E$ for a $\mu_D \in \mathbb{Z}$, the multiplicity of the point P in the variety D .

Claim 4.2. $\mu_D = dH^3$.

Proof. Since H is very ample, by Bertini there exist two smooth hyperplane sections $H_1, H_2 \in |H|$ with $H_1 \cap H_2 \cap D = \{P\}$. Then $H_1 \cap H_2$ does not contain any line L from P to a point $Q \in C$.

As in the proof of the previous claim the H_i intersect every line L from P to a point $Q \in C - H_i$ exactly in P with intersection multiplicity 1, i.e. transversal. Therefore the strikt transforms $\tilde{D}, \tilde{H}_1, \tilde{H}_2$ do not intersect at all. Since the H_i are smooth, the intersection multiplicities of the H_i in P are $\mu_{H_i} = 1$. Then, by [Ful84, 12.4.8]

$$\mu_D = \mu_D \mu_{H_1} \mu_{H_2} = D.H_1.H_2 = dH^3.$$

□

The strict transform S of the cone D is isomorphic to the \mathbb{P}^1 -bundle

$$S \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(C))$$

over the curve C . Let $f : \widehat{C} \rightarrow C$ the normalization of the possibly singular curve C and let the \mathbb{P}^1 -bundle

$$\widehat{S} \cong \mathbb{P}(\mathcal{O}_{\widehat{C}} \oplus f^* \mathcal{O}_C(C))$$

over \widehat{C} be the normalization of S .

Thus one has the following diagram:

$$\begin{array}{ccccc} & & \tilde{X} & \xrightarrow{\pi} & X \\ & & \uparrow & & \uparrow \\ \hat{S} & \xrightarrow{f} & S & \xrightarrow{\pi} & D \\ \downarrow \hat{p} & & \downarrow p & & \\ \hat{C} & \xrightarrow{f} & C & & \end{array}$$

Claim 4.3. $H^3 \leq 4$.

Proof. By the adjunction formulas one gets

$$-K_{X|D} = -K_D + D|_D$$

and

$$\begin{aligned} K_S &= K_{\tilde{X}|S} + S|_S \\ &= \pi^*(K_X)|_S + 2E|_S + \pi^*(D)|_S - \mu E|_S = \\ &= \pi^*(K_{X|D} + D|_D) + (2 - \mu)E|_S = \pi^*(K_D) + (2 - \mu)E|_S. \end{aligned}$$

The nonnormal locus N on S is given by the conductor ideal of the normalization $f : \widehat{S} \rightarrow S$, and its support consists of whole fibers of the \mathbb{P}^1 -bundle $S \rightarrow C$. From the subadjunction formula for normalization (and the formula for the canonical bundle on smooth ruled surfaces), it follows

$$f^* K_S = K_{\widehat{S}} - N = -2C_0 + kF, \quad k \in \mathbb{Z},$$

where $C_0 \cong \widehat{C}$ is the section with negative self intersection on the smooth ruled surface \widehat{S} and F is a fiber.

Let $F_{\widehat{S}}$ be a fiber of $\widehat{p} : \widehat{S} \rightarrow \widehat{C}$ and F_S a general fiber of $p : S \rightarrow C$. Then $f_* F_{\widehat{S}} = F_S$. Furthermore, $\pi_* F_S = L$, where L is a line from the cone vertex P to a point $Q \in C$. Since $E|_S$ is a section of the projective line bundle S over C , the intersection multiplicity $F_S \cdot E|_S = 1$. Therefore:

$$\begin{aligned} -2 &= (-2C_0 + kF) \cdot F = f^* K_S \cdot F_{\widehat{S}} = K_S \cdot F_S = \\ &= \pi^*(K_D) \cdot F_S + (2 - \mu)E|_S \cdot F_S = K_D \cdot \pi_* F_S + 2 - H^3 d = \\ &= K_D \cdot L + 2 - H^3 d. \end{aligned}$$

Let r_D be defined by $-K_D = -r_D H|_D$. As in the proof of Claim 4.1, $H|_D \cdot L = 1$. Consequently:

$$-2 = -r_D H|_D \cdot L + 2 - H^3 d = -r_D + 2 - H^3 d,$$

and this implies $r_D = 4 - H^3 d$. But the index r_X of the Fano threefold X is ≥ 1 , hence

$$r_X = r_D + d = 4 - (H^3 - 1)d \geq 1,$$

and $H^3 \leq 4$. □

Now, the following cases must be considered:

- $H^3 = 4$: Then $d = 1$ and $r_X = 1$. Under the assumption that H is very ample, X must be a quartic.
- $H^3 = 3$: Then still $d = 1$, but $r_X = 2$. So X must be a cubic.
- $H^3 = 2$: Then $d = 1, 2$ or 3 , corresponding to $r_X = 3, 2, 1$, and X must be a quadric (if H is very ample).
- $H^3 = 1$: This implies $r_X = 4$ and $X = \mathbb{P}^3$.

5. SPECIAL FANOS

In this section, vector fields vanishing on the reduced and irreducible intersection of two (very) ample divisors on \mathbb{P}^3 and the quadric Q_3 will be constructed, and it will be shown that such vector fields do not exist on a cubic, on a quartic and on varieties of type V_1, V_2, V'_2, V''_4 , where H is not very ample.

5.1. V_1, V_2, V'_2 und V''_4 . These Fano varieties are described in [Isk78]:

- (a) The morphism $\varphi_{K_{V_1}^{-1}} : V_1 \rightarrow W_4 \subset \mathbb{P}^6$ induced by the complete linear system $|K_{V_1}^{-1}|$ is a 2:1-covering of the cone W_4 over the Veronese surface $F_4 \subset \mathbb{P}^5$. It branches over the smooth divisor $S \subset W_4$ cut out by a cubic hypersurface not containing the cone vertex.
- (b) The morphism $\varphi_H : V_2 \rightarrow \mathbb{P}^3$ is a 2:1-covering with smooth branching divisor $S \subset \mathbb{P}^3$ of degree 4.
- (c) The morphism $\varphi_H : V'_2 \rightarrow \mathbb{P}^3$ is a 2:1-covering with smooth branching divisor $S \subset \mathbb{P}^3$ of degree 6.
- (d) The morphism $\varphi_H : V''_4 \rightarrow Q_3 \subset \mathbb{P}^4$ is a 2:1-covering of the three-dimensional quadric with smooth branching divisor $S \subset Q_3$ cut out by a hypersurface of degree 4.

Assume that on one of these Fano threefolds there exists a vector field $v \in H^0(X, T_X)$ vanishing on a curve. Then there is a linear algebraic group G acting on X and inducing v . Since the coverings are induced by complete linear systems, G acts on the bases W of the coverings, too. Furthermore G stabilizes the branching divisors $S \in W$ since these divisors describe the locus of the points where the rank of the differentials $df_x : T_{X,x} \rightarrow T_{W,f(x)}$ drops.

Therefore, v restricted to S is a (non trivial) vector field on the branching divisor. By assumption, v should vanish on a curve C . Since the branching divisor S is ample, S intersects C , and $v|_S$ has a zero. As explained in section 3 this implies the existence of a \mathbb{C}^+ - or \mathbb{C}^* -action on S . Consequently, S is covered by rational curves, i.e. the closures of the orbits. S is uniruled, and K_S is not nef.

On the other hand it is easy to compute K_S since the branching divisors are complete intersections:

- (a) On varieties of type V_1 is $K_S = \mathcal{O}_S(2)$, because the Veronese surface and the cone over it are generated by three quadrics.
- (b) If X is of type V_2 , then $K_S = \mathcal{O}_S$.
- (c) X of type V'_2 : then $K_S = \mathcal{O}_S(2)$.

(d) X of type V_4'' : then $K_S = \mathcal{O}_S(1)$.

In all cases K_S is nef, contradiction.

5.2. Quadrics, cubics and quartics. The starting point is the following

Lemma 5.1. *All vector fields on a hypersurface $H \in \mathbb{P}^n$, $n > 2$, are induced by equivariant vector fields on \mathbb{P}^n .*

Proof. Set $d := \deg H$. First consider the structure sheaf sequence of H tensorized by the tangent bundle $T_{\mathbb{P}^n}$,

$$0 \rightarrow T_{\mathbb{P}^n}(-d) \rightarrow T_{\mathbb{P}^n} \rightarrow T_{\mathbb{P}^n} \otimes \mathcal{O}_H \rightarrow 0,$$

and the beginning of the corresponding long exact sequence,

$$0 \rightarrow H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-d)) \rightarrow H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \xrightarrow{\nu} H^0(H, T_{\mathbb{P}^n} \otimes \mathcal{O}_H) \rightarrow H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-d)).$$

The Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1-d)^{n+1} \rightarrow T_{\mathbb{P}^n}(-d) \rightarrow 0$$

and $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $0 < i < n$, $k \in \mathbb{Z}$ imply $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-d)) = 0$. Therefore, ν is surjective. By the normal sequence

$$0 \rightarrow \mathcal{T}_H \rightarrow T_{\mathbb{P}^n} \otimes \mathcal{O}_H \rightarrow N_{H/\mathbb{P}^n} \rightarrow 0$$

one finally has $H^0(H, \mathcal{T}_H) \subset H^0(H, T_{\mathbb{P}^n} \otimes \mathcal{O}_H)$. \square

From now on, let $H \subset \mathbb{P}^4$ be a smooth hypersurface of degree $d = 2, 3, 4$ with a vector field $v \in H^0(H, \mathcal{T}_H)$ vanishing on the irreducible and reduced intersection $C = H_1 \cap H_2$ of two very ample divisors. The lemma above shows that v is induced by a vector field on \mathbb{P}^4 also called v .

Claim 4.1 and the cases at the end of section 4 show that v stabilizes a cone $D = H \cap H_1$ cut out by a hyperplane, and the zero locus of v is the intersection $C = H \cap H_1 \cap H_2$ with another hyperplane. The cone vertex $P \in D$ is not contained in H_2 since otherwise, $D \cap H_2$ contains more than one line because of $\deg H > 1$.

Now choose homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ on \mathbb{P}^4 such that the cone D lies on the hyperplane $H_1 = \{x_4 = 0\}$, and the cone basis lies in the plane $H_1 \cup H_2 = \{x_3 = x_4 = 0\}$. Furthermore, let $P = (0 : 0 : 0 : 1 : 0)$.

Lemma 5.2. *The hypersurface H is given by a polynomial*

$$h = f(x_0, x_1, x_2) + x_4 g(x_0, x_1, x_2, x_3, x_4),$$

where $\deg f = d$, $\deg g = d - 1$. The coefficients of the monomial x_3^{d-1} in g and of the monomial $x_i x_4^{d-1}$ in h do not vanish (at least for one $0 \leq i \leq 4$).

Proof. The polynomial h may be written as

$$h = f(x_0, x_1, x_2) + x_3 k(x_0, x_1, x_2, x_3) + x_4 g(x_0, x_1, x_2, x_3, x_4).$$

Since the intersection with $H_1 = \{x_4 = 0\}$ is supposed to be a cone over a basis $C \subset \{x_3 = x_4 = 0\}$ with vertex $(0 : 0 : 0 : 1 : 0)$, this cone D is given by the equation $f(x_0, x_1, x_2) = 0$, and $k \equiv 0$.

Since the hypersurface is smooth, the rest follows from computing the gradient of h : the coefficient of x_3^{d-1} must not be 0 because otherwise the vertex $(0 : 0 : 0 : 1 : 0)$ will be a singularity in H , too. Similarly, the coefficient of at least one of the monomials $x_i x_4^{d-1}$, $i \leq 4$, must not vanish, because otherwise the point $(0 : 0 : 0 : 0 : 1)$ lies in H and will be not smooth. \square

Now, by the Euler sequence vector fields on \mathbb{P}^4 correspond to homogeneous derivations $D = \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j}$ of weight 0 on $\mathbb{C}[x_0, \dots, x_4]$, modulo the Euler vector fields $\sum_i a x_i \frac{\partial}{\partial x_i}$.

Lemma 5.3. *The vector field v corresponds to the derivation $D_v = \sum_{i,j} a_{ij} x_i \frac{\partial}{\partial x_j}$ given by (a non trivial scalar multiple of) the matrix*

$$(a_{ij}) = \left(\begin{array}{cccc|c} 0 & \cdots & \cdots & 0 & 0 \\ \vdots & 0 & & \vdots & \vdots \\ \vdots & & 0 & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ \hline * & * & * & * & a \end{array} \right).$$

Proof. Since $\deg C > 1$, the curve C is no line. Since vector fields on projective spaces always vanish on complete linear subspaces, v vanishes on the plane $x_3 = x_4 = 0$. On the other hand, v does not vanish on the hyperplane $x_4 = 0$, because it contains the cone D , and v should be non trivial on D . Since every vector field on a cone vanishes in the vertex, v vanishes in $P = (0 : 0 : 0 : 1 : 0)$.

This implies the hypothesis, because the zero locus of vector fields on \mathbb{P}^4 consists of the eigenspaces of the transposed corresponding matrix. \square

Now, v stabilizes the hypersurface H exactly when the derivation D_v maps the principal ideal $(h) \subset \mathbb{C}[x_0, \dots, x_4]$ describing H to itself, i.e. $D_v h = \lambda h$ for a $\lambda \in \mathbb{C}$. Since $a_{ij} = 0$ for $i, j = 0, 1, 2$,

$$D_v h = D_v f + D_v x_4 \cdot g + x_4 D_v g = 0 + x_4(ag + D_v g),$$

and consequently $\lambda = 0$.

This gives immediately a quadric with a vector field vanishing as in Theorem 1.2: Let $h = x_0^2 + x_1^2 + x_2^2 + x_3 x_4$ be the equation of the matrix and let v correspond to the derivation $D_v = x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}$. Then, $D_v h = 0$, and v vanishes on the smooth quadric $x_0^2 + x_1^2 + x_2^2 = 0$ contained in the plane $x_3 = x_4 = 0$ and in the cone vertex $(0 : 0 : 0 : 1 : 0)$. Furthermore, v stabilizes the cone $x_0^2 + x_1^2 + x_2^2 = 0$ contained in the hyperplane $x_4 = 0$.

Why are there no such vector fields on cubics and quartics ?

- (a) By Lemma 5.2 the monomial $x_3^{d-1} x_4$ has a coefficient $c \neq 0$ in h , but there is no monomial of the form $x_3^{d-1} x_i$, $i \leq 3$, in h .
- (b) The coefficient of $x_3^{d-1} x_4$ in $D_v h = 0$ is 0. Decomposing h in monomials and D_v in “monomial” derivations of the form $a_{ij} x_i \frac{\partial}{\partial x_j}$, one sees that only the derivations

$$a_{3j} x_3 \frac{\partial}{\partial x_j} x_j x_3^{d-2} x_4, \quad a_{4j} x_4 \frac{\partial}{\partial x_j} x_j x_3^{d-1} x_j$$

contribute to the coefficient of $x_3^{d-1} x_4$. Since $a_{3j} = 0$ for $i = 0, 1, 2, 4$ and the coefficients of $x_3^{d-1} x_i$, $i \leq 3$, vanish, the coefficient of $x_3^{d-1} x_4$ in $D_v h$ is the coefficient of

$$c(a_{33} x_3 \frac{\partial}{\partial x_3} x_3^{d-1} x_4 + a_{44} x_4 \frac{\partial}{\partial x_4} x_3^{d-1} x_4),$$

i.e. $c(d - 1 + a)$.

- (c) Consequently, $a = 1 - d$. This number is an eigenvalue of the matrix corresponding to the derivation, different from the other Eigen values. Therefore, in appropriate coordinates the matrix is diagonal, i.e.

$$D_v = x_3 \frac{\partial}{\partial x_3} + (1 - d)x_4 \frac{\partial}{\partial x_4}.$$

- (d) The difference between quadrics and cubics resp. quartics comes from the fact that the monomials $x_3x_4^{d-1}$ and $x_3^{d-1}x_4$ are different for $d > 2$. The coefficient c of $x_3x_4^{d-1}$ in h must be 0 for $d > 2$, because otherwise the coefficient of $x_3x_4^{d-1}$ in Dh equals $c(1 - (d-1)^2) \neq 0$. The same is true for the monomials $x_ix_4^{d-1}$ with $i = 0, 1, 2, 4$. But this is a contradiction to Lemma 5.2.

Remark 5.4. There is another argumentation for cubics: The Fano variety $F(V_3)$ of the lines on the cubic V_3 is a smooth variety with a very ample canonical divisor ([CG72, 7.8, 10.13]). Furthermore, through a general point there are exactly 6 lines ([Tju72]).

But as already shown in the beginning of the section, the existence of a vector field with zeroes implies an effective linear algebraic group operation with fixed points on V_3 . Every group operation on V_3 induces a group operation on $F(V_3)$, and this operation must be trivial because of the very ample canonical divisor. Therefore, lines on V_3 are stabilized by the operation, and the intersection point of 6 lines is fixed. The operation on V_3 is trivial, too.

5.3. The projective space \mathbb{P}^3 . A vector field vanishing on a line is given by $D_v = x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$.

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VECTOR FIELDS ON SMOOTH THREEFOLDS VANISHING ON COMPLETE INTERSECTIONS

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